

Exposed positive maps: a sufficient condition

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Abstract

Exposed positive maps in matrix algebras define a dense subset of extremal maps. We provide a sufficient condition for a positive map to be exposed. This is an analog of a spanning property which guaranties that a positive map is optimal. We analyze a class of decomposable maps for which this condition is also necessary.

1 Introduction

Positive maps in \mathbb{C}^* -algebras play an important role both in mathematics, in connection with the operator theory [1], and in modern quantum physics. Normalized positive maps provide an affine mapping between sets of states of \mathbb{C}^* -algebras. In recent years positive maps found important application in entanglement theory defining basic tool for detecting quantum entangled states (see e.g. [2] for the recent review).

Let \mathfrak{U} be a unital \mathbb{C}^* -algebra. A linear map $\Phi : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H})$ is positive if $\Phi(\mathfrak{U}_+) \subset \mathcal{B}_+(\mathcal{H})$, where \mathfrak{U}_+ denotes positive elements in \mathfrak{U} . Denote by $M_k(\mathfrak{U}) = M_k(\mathbb{C}) \otimes \mathfrak{U}$ a space of $k \times k$ matrices with entries from \mathfrak{U} . One says that Φ is k -positive if a linear map $\Phi_k := \mathbb{1}_k \otimes \Phi : M_k(\mathfrak{U}) \rightarrow M_k(\mathcal{B}(\mathcal{H}))$ is positive. Finally, Φ is completely positive if it is k -positive for $k = 1, 2, \dots$. Due to the Stinespring theorem [3] the structure of completely positive maps is perfectly known: any completely positive map Φ may be represented in the following form

$$\Phi(a) = V^\dagger \pi(a) V, \quad (1)$$

where $V : \mathcal{H} \rightarrow \mathcal{K}$, and π is a representation of \mathfrak{U} in the Hilbert space \mathcal{K} . Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood [4]–[23].

Denote by \mathcal{P} a convex cone of positive maps $\Phi : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H})$. Note, that a space $\mathcal{L}(\mathfrak{U}, \mathcal{B}(\mathcal{H}))$ of linear maps from \mathfrak{U} to $\mathcal{B}(\mathcal{H})$ is isomorphic to $\mathcal{B}(\mathcal{H}) \otimes \mathfrak{U}$. The natural pairing between these two spaces is defined as follows [13]: taking an orthonormal basis in \mathcal{H} ($m = \dim \mathcal{H} < \infty$) one identifies $\mathcal{B}(\mathcal{H})$ with $M_m(\mathbb{C})$ and defines

$$\langle X \otimes a, \Phi \rangle := \text{tr}(X^t \Phi(a)), \quad (2)$$

where $X \in M_m(\mathbb{C})$, $a \in \mathfrak{U}$, and X^t denotes transposition of X with respect to a given basis in \mathcal{H} . Let \mathcal{P}° denote a dual cone [13, 24]

$$\mathcal{P}^\circ = \{ A \in M_m(\mathbb{C}) \otimes \mathfrak{U} : \langle A, \Phi \rangle \geq 0, \Phi \in \mathcal{P} \}. \quad (3)$$

Note that the definition of \mathcal{P}° may be reformulated as follows

$$\mathcal{P}^\circ = \text{conv}\{ X \otimes a \in M_m(\mathbb{C}) \otimes \mathfrak{U} : \langle X \otimes a, \Phi \rangle \geq 0, \Phi \in \mathcal{P} \}. \quad (4)$$

One finds $\mathcal{P}^\circ = M_m^+(\mathbb{C}) \otimes \mathfrak{U}_+$, where $M_m^+(\mathbb{C})$ denotes positive matrices from $M_m(\mathbb{C})$. It shows that \mathcal{P}° defines a convex cone of separable elements in $M_m(\mathbb{C}) \otimes \mathfrak{U}$.

Recall that a face of \mathcal{P} is a convex subset $F \subset \mathcal{P}$ such that if the convex combination $\Phi = \lambda\Phi_1 + (1-\lambda)\Phi_2$ of $\Phi_1, \Phi_2 \in \mathcal{P}$ belongs to F , then both $\Phi_1, \Phi_2 \in F$. If a ray $\{\lambda\Phi : \lambda > 0\}$ is a face of \mathcal{P} then it is called an extreme ray, and we say that Φ generates an extreme ray. For simplicity we call such Φ an extremal positive map. A face F is exposed if there exists a supporting hyperplane H for a convex cone \mathcal{P} such that $F = H \cap \mathcal{P}$. The property of ‘being an exposed face’ may be reformulated as follows: A face F of \mathcal{P} is exposed iff there exists $a \in \mathfrak{U}_+$ and $|h\rangle \in \mathcal{H}$ such that

$$F = \{ \Phi \in \mathcal{P} ; \Phi(a)|h\rangle = 0 \}.$$

A positive map $\Phi \in \mathcal{P}$ is exposed if it generates 1-dimensional exposed face. Let us denote by $\text{Ext}(\mathcal{P})$ the set of extremal points and $\text{Exp}(\mathcal{P})$ the set of exposed points of \mathcal{P} . Due to Straszewicz theorem [24] $\text{Exp}(\mathcal{P})$ is a dense subset of $\text{Ext}(\mathcal{P})$. Thus every extreme map is the limit of some sequence of exposed maps meaning that each entangled state may be detected by some exposed positive map. Hence, a knowledge of exposed maps is crucial for the full characterization of separable/entangled states of bi-partite quantum systems. For recent papers on exposed maps see e.g. [13, 21, 22, 23].

Now, if F is a face of \mathcal{P} then

$$F' = \text{conv}\{ a \otimes |h\rangle\langle h| \in \mathcal{P}^\circ : \Phi(a)|h\rangle = 0, \Phi \in F \}. \quad (5)$$

defines a face of \mathcal{P}° (one calls F' a dual face of F). Actually, F' is an exposed face. One proves [13] that a face F is exposed iff $F'' = F$.

In this paper we analyze linear positive maps $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$, where both \mathcal{K} and \mathcal{H} are finite dimensional Hilbert spaces. We provide a sufficient condition for the map to be exposed. We call it *strong spanning property* in analogy to well known spanning property which is sufficient for the map to be optimal [26]. Interestingly, this condition is also necessary if Φ is decomposable and $\dim \mathcal{K} = 2$. Finally, we characterize the property of exposedness in terms of entanglement witnesses.

2 Preliminaries

Consider a positive map $\Phi : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H})$, where \mathfrak{U} is a unital \mathbb{C}^* -algebra and $\mathcal{B}(\mathcal{H})$ denotes a set of bounded operators on the finite dimensional Hilbert space \mathcal{H} .

Proposition 2.1 *If $a \in \mathfrak{U}$ is strictly positive, i.e. $a \in \text{int } \mathfrak{U}_+$, then $\text{Im } \Phi(b) \subset \text{Im } \Phi(a)$ for all $b \in \mathfrak{U}_+$.*

Proof: Let us observe that

$$\text{Ker } \Phi(a) \subset \text{Ker } \Phi(b). \quad (6)$$

Indeed, suppose that there exists $x \in \mathcal{H}$ such that $x \in \text{Ker } \Phi(a)$ and $x \notin \text{Ker } \Phi(b)$. One has $\langle x | \Phi(b) | x \rangle > 0$ and $\langle x | \Phi(a) | x \rangle = 0$. Now, since $a \in \text{int } \mathfrak{U}_+$ there exists $\epsilon > 0$ such that an open ball $B(a, \epsilon) \subset \mathfrak{U}_+$. It is therefore clear that

$$a' = a - \frac{\epsilon}{2} \frac{u - a}{\|u - a\|}$$

belongs to \mathfrak{U}_+ . One has

$$\langle x | \Phi(a') | x \rangle = -\frac{\epsilon}{2\|u - a\|} \langle x | \Phi(u) | x \rangle < 0, \quad (7)$$

which contradicts that Φ is a positive map. Hence, if $a \in \text{int } \mathfrak{U}_+$, then $\text{Ker } \Phi(a) \subset \text{Ker } \Phi(b)$ for any $b \in \mathfrak{U}_+$ which implies $\text{Im } \Phi(b) \subset \text{Im } \Phi(a)$. \square .

Corollary 2.1 *If $a, b \in \text{int } \mathfrak{U}_+$, then $\text{Im } \Phi(a) = \text{Im } \Phi(b)$.*

Corollary 2.2 *In particular for $a \in \mathfrak{U}_+$ ($a \in \text{int } \mathfrak{U}_+$), one has $\text{Im } \Phi(a) \subset \text{Im } \Phi(\mathbb{1})$ ($\text{Im } \Phi(a) = \text{Im } \Phi(\mathbb{1})$).*

Let $A := \Phi(\mathbb{1})$. If $A > 0$, that is, A is of full rank, then one has

$$\Phi(a) = A^{1/2} \tilde{\Phi}(a) A^{1/2}, \quad (8)$$

where $\tilde{\Phi}(a) = A^{-1/2} \Phi(a) A^{-1/2}$ is a unital positive map from \mathfrak{U} into $\mathcal{B}(\mathcal{H})$. If A is not strictly positive, that is, $A \in \partial \mathcal{B}_+(\mathcal{H})$, then denote by \mathcal{H}_Φ the range of A . A is invertible on its image and denote by \tilde{A}^{-1} the generalized inverse of A . Now, one has

$$\Phi(a) = A^{1/2} \tilde{A}^{-1/2} \Phi(a) \tilde{A}^{-1/2} A^{1/2}. \quad (9)$$

Note, that $\text{Im } \Phi(a) \subset \mathcal{H}_\Phi$. Following [8] let us introduce the following

Definition 2.1 *Consider a positive map $\phi : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H})$. A map $\phi' : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H}')$ is called an extension of ϕ iff $\mathcal{H}' \supset \mathcal{H}$ and for any $a \in \mathfrak{U}$*

$$\phi(a) = \mathcal{P} \phi'(a) \mathcal{P}, \quad (10)$$

where \mathcal{P} denotes orthogonal projection $\mathcal{H}' \rightarrow \mathcal{H}$.

Note that $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}^\perp$ and hence for any $|h'\rangle \in \mathcal{H}'$ one has $|h'\rangle = |h\rangle \oplus |h^\perp\rangle$, where $|h\rangle = \mathcal{P}|h'\rangle$ which implies $\phi(a)|h\rangle = \mathcal{P}\phi'(a)|h\rangle$, and an extension Φ' is trivial if

$$\phi'(a)|h\rangle = \phi(a)|h\rangle, \quad (11)$$

for all $a \in \mathfrak{U}$ and $|h\rangle \in \mathcal{H}$. According to this definition a positive map $\tilde{\Phi}(a) := \tilde{A}^{-1/2} \Phi(a) \tilde{A}^{-1/2}$ is a trivial extension of the unital map $\Phi_1 : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H}_\Phi)$

$$\Phi_1 = \mathcal{P}_\Phi \tilde{\Phi} \mathcal{P}_\Phi, \quad (12)$$

where \mathcal{P}_Φ is a projector $\mathcal{H} \rightarrow \mathcal{H}_\Phi$. This way we proved the following

Proposition 2.2 Any linear positive map $\Phi : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H})$ can be written as follows

$$\Phi(a) = V^\dagger \Phi_1(a) V , \quad (13)$$

where $V : \mathcal{H} \rightarrow \mathcal{H}_\Phi$ and $\Phi_1 : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H}_\Phi)$ is unital.

Let us recall

Definition 2.2 A linear map Φ is irreducible if $[\Phi(a), X] = 0$ for all $a \in \mathfrak{U}$ implies that $X = \lambda \mathbb{I}_\mathcal{H}$. Φ is irreducible on its image if $[\Phi(a), X] = 0$ for all $a \in \mathfrak{U}$ implies that $\mathcal{P}_\Phi X \mathcal{P}_\Phi = \lambda \mathbb{I}_{\mathcal{H}_\Phi}$.

Remark 2.1 Note, that one may restrict oneself to self-adjoint elements \mathfrak{U}_{sa} only. Indeed, suppose that Φ is irreducible and $[\Phi(a), X] = 0$ for all $a \in \mathfrak{U}_{\text{sa}}$. Any element $x \in \mathfrak{U}$ may be decomposed as $x = x_1 + ix_2$, with $x_1, x_2 \in \mathfrak{U}_{\text{sa}}$. One has

$$[\Phi(x), X] = [\Phi(x_1), X] + i[\Phi(x_2), X] = 0 ,$$

and irreducibility of Φ implies therefore $X = \lambda \mathbb{I}_{\mathcal{H}_\Phi}$.

Proposition 2.3 Let a positive map Φ be irreducible. If $X\Phi(a) = \Phi(a)X^\dagger$ for all $a \in \mathfrak{U}$, then $X = \lambda \mathbb{I}_\mathcal{H}$.

Proof: Irreducibility implies that $A = \Phi(\mathbb{1}) > 0$ and hence

$$\Phi(a) = A^{1/2} \Phi_1(a) A^{1/2} , \quad (14)$$

where Φ_1 is unital. One has

$$X A^{1/2} \Phi_1(a) A^{1/2} = A^{1/2} \Phi_1(a) A^{1/2} X^\dagger ,$$

and hence

$$Y \Phi_1(a) = \Phi_1(a) Y^\dagger , \quad (15)$$

with $Y = A^{-1/2} X A^{1/2}$. Using $\Phi_1(\mathbb{1}) = \mathbb{I}_\mathcal{H}$ one finds $Y^\dagger = Y$. Let us observe that Φ_1 is irreducible as well and hence $Y = \lambda \mathbb{I}_\mathcal{H}$ which implies $X = \lambda \mathbb{I}_\mathcal{H}$. \square

3 Exposed maps – sufficient condition

In this section we formulate a sufficient condition for a map $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$ to be exposed. Recall that a linear operator $W \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ is block-positive iff $\langle x \otimes y | W | x \otimes y \rangle \geq 0$ for all product vectors $|x \otimes y\rangle \in \mathcal{K} \otimes \mathcal{H}$. Now, due to the Choi-Jamiołkowski isomorphism, W is block-positive iff there exists a positive map $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$ such that

$$W = (\mathbb{I}_\mathcal{K} \otimes \Phi) P_\mathcal{K}^+ ,$$

where $\mathbb{I}_\mathcal{K}$ is an identity map in $\mathcal{B}(\mathcal{K})$, and $P_\mathcal{K}^+$ is a maximally entangled state in $\mathcal{K} \otimes \mathcal{K}$. Any block-positive but not positive W is called an entanglement witness. It is therefore clear that any property of a map Φ may be formulated in terms of W and vice versa. Now, let us define

$$P_W = \{ x \otimes y : \langle x \otimes y | W | x \otimes y \rangle = 0 \} . \quad (16)$$

Note, that

$$\langle x \otimes y | W | x \otimes y \rangle = \langle y | \Phi(|\bar{x}\rangle\langle\bar{x}|) | y \rangle ,$$

and hence one may equivalently introduce $P_\Phi \equiv P_W = \{ x \otimes y : \Phi(|\bar{x}\rangle\langle\bar{x}|) | y \rangle = 0 \}$. One says that Φ has *spanning property* iff $\text{span}_{\mathbb{C}} P_\Phi = \mathcal{K} \otimes \mathcal{H}$. Denoting $d_{\mathcal{K}} = \dim \mathcal{K}$ and $d_{\mathcal{H}} = \dim \mathcal{H}$, one proves [26]

Theorem 3.1 *If a positive map Φ satisfies spanning property, then it is optimal.*

In analogy we have the following

Theorem 3.2 *Let $\Phi : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a positive map irreducible on its image and*

$$N_\Phi = \text{span}_{\mathbb{C}} \{ a \otimes |h\rangle \in \mathcal{B}_+(\mathcal{K}) \otimes \mathcal{H} : \Phi(a)|h\rangle = 0 \} . \quad (17)$$

If the subspace $N_\Phi \subset \mathcal{B}(\mathcal{K}) \otimes \mathcal{H}$ satisfies

$$\dim N_\Phi = d_{\mathcal{K}}^2 d_{\mathcal{H}} - \text{rank } \Phi(\mathbb{I}_{\mathcal{K}}) , \quad (18)$$

then Φ is exposed.

Proof: The idea of the proof comes from [8] (see Theorem 3.3). Consider a map [25]

$$\tilde{\Phi} : \mathcal{B}(\mathcal{K}) \otimes \mathcal{H} \longrightarrow \mathcal{H}$$

defined by

$$\tilde{\Phi}(a \otimes |h\rangle) := \Phi(a)|h\rangle . \quad (19)$$

Note, that $\dim(\text{Im } \tilde{\Phi}) = \text{rank } \Phi(\mathbb{I}_{\mathcal{K}})$ and hence N_Φ defines the kernel of $\tilde{\Phi}$. To show that Φ is exposed let us introduce a linear functional f on the space of positive maps $\mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$ defined as follows

$$f(\Psi) = \sum_{i=1}^{d_N} \langle h_i | \Psi(a_i) | h_i \rangle , \quad (20)$$

where d_N vectors $a_i \otimes |h_i\rangle$ span N_Φ . Note that $f(\Psi) \geq 0$ for all positive maps Ψ and $f(\Phi) = 0$. As a result f defines a supporting hyperplane to the cone of positive maps $\mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$ passing through a map Φ . Note that Φ is exposed iff $f(\Psi) = 0$ implies $\Psi = \lambda\Phi$, with λ being a positive number. Let us observe that $f(\Psi) = 0$ if and only if $\tilde{\Psi}(a_i \otimes |h_i\rangle) = \Psi(a_i)|h_i\rangle = 0$, for all $i = 1, \dots, d_N$, and hence the kernel of $\tilde{\Psi}$ contains N_Φ . To complete the proof we use the following

Lemma 3.1 *Consider two linear operators $A, B : V \rightarrow W$, where V and W are finite dimensional vector spaces over \mathbb{C} . If $\ker A \supset \ker B$, then there exists $X : W \rightarrow W$ such that $A = XB$ and $\text{rank } X = \text{rank } A$.*

Proof: let

$$A = U_A \Sigma_A V_A^\dagger , \quad B = U_B \Sigma_B V_B^\dagger ,$$

denote the corresponding singular value decompositions of A and B . Let $\{v_\alpha(A)\}$, $\{w_\alpha(A)\}$, $\{v_\alpha(B)\}$ and $\{w_\alpha(B)\}$ denote the orthonormal basis made from columns of V_A , U_A , V_B , U_B respectively. One has

$$\Sigma_A = \sum_{\alpha=1}^{r_A} \sigma_\alpha(A) |w_\alpha(A)\rangle \langle v_\alpha(A)| , \quad \Sigma_B = \sum_{\alpha=1}^{r_B} \sigma_\alpha(B) |w_\alpha(B)\rangle \langle v_\alpha(B)| , \quad (21)$$

where $\sigma_\alpha(A)$ and $\sigma_\alpha(B)$ are strictly positive singular values of A and B , respectively. Note, that condition $\ker A \supset \ker B$, is equivalent to $r_B \geq r_A$. One finds $A = XB$, where

$$X = AV_B^\dagger \tilde{\Sigma}_B U_B^\dagger , \quad (22)$$

with

$$\tilde{\Sigma}_B = \sum_{\alpha=1}^{r_B} \sigma_\alpha(B)^{-1} |w_\alpha\rangle \langle v_\alpha| . \quad (23)$$

Indeed, one has

$$XB = (AV_B^\dagger \tilde{\Sigma}_B U_B^\dagger)(U_B \Sigma_B V_B^\dagger) = AV_B^\dagger \tilde{\Sigma}_B \Sigma_B V_B^\dagger = A \sum_{\alpha=1}^{r_B} |v_\alpha(B)\rangle \langle v_\alpha(B)| .$$

Now, since $\ker A \supset \ker B$, one has

$$A \sum_{\alpha=1}^{r_B} |v_\alpha(B)\rangle \langle v_\alpha(B)| = A ,$$

which ends the proof. \square

One has, therefore, $\tilde{\Psi} = X\tilde{\Phi}$, for some operator X acting on the image of $\tilde{\Phi}$, meaning that

$$\Psi(a)|h\rangle = X\Phi(a)|h\rangle ,$$

for all $a \in \mathcal{B}(\mathcal{K})$ and $|h\rangle \in \mathcal{H}$. Note that for any $a \in \mathcal{B}_{\text{sa}}(\mathcal{K})$ one has $\Psi(a) = \Psi(a)^\dagger$ and hence

$$X\Phi(a) = \Phi(a)X^\dagger . \quad (24)$$

Proposition 2.3 implies, therefore, that $X \sim \mathbb{I}$ on the image of Φ . Hence $\Psi = \lambda\Phi$ with $\lambda > 0$ due to the fact that both Φ and Ψ are positive maps. \square

Corollary 3.1 *Let $\Phi : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a positive, unital irreducible map. If*

$$\dim N_\Phi = (d_{\mathcal{K}}^2 - 1)d_{\mathcal{H}} , \quad (25)$$

then Φ is exposed.

We propose to call (18) *strong spanning property* in analogy to spanning property

$$\dim \text{span}_{\mathbb{C}}\{|x\rangle \otimes |h\rangle \in \mathcal{K} \otimes \mathcal{H} : \Phi(|\bar{x}\rangle \langle \bar{x}|)|h\rangle = 0\} = d_{\mathcal{K}}d_{\mathcal{H}} , \quad (26)$$

which is sufficient for optimality.

4 A class of exposed decomposable maps $\mathcal{B}(\mathbb{C}^2) \longrightarrow \mathcal{B}(\mathbb{C}^m)$

In this section we provide a class of positive exposed maps for which strong spanning property (18) is also necessary.

Theorem 4.1 *Let $\Phi : \mathcal{B}(\mathbb{C}^2) \rightarrow \mathcal{B}(\mathbb{C}^m)$ be a decomposable positive but not completely positive map. Then the following conditions are equivalent:*

1. Φ is exposed.
2. $\Phi(\rho) = V^\dagger \rho^\dagger V$, where $V : \mathbb{C}^n \rightarrow \mathbb{C}^2$ is a linear map of rank two.
3. There are $4m - 2$ linearly independent vectors in the set $\{a \otimes |h\rangle \in \mathcal{B}_+(\mathbb{C}^2) \otimes \mathbb{C}^m : \Phi(a)|h\rangle = 0\}$.

Proof: (1 \Rightarrow 2) Any exposed map is extremal and hence being a decomposable map Φ is given by $\Phi(a) = V^\dagger a V$ or $\Phi(a) = V^\dagger a^\dagger V$. The former is evidently CP and the latter is not CP iff $\text{rank}(V) = 2$.

(2 \Rightarrow 3) Note, that using linear transformation one can transform V to the following form $V = \sum_{i=1}^2 |e_i\rangle\langle f_i|$, where $\{e_i\}_{i=1}^2, \{f_j\}_{j=1}^m$ are orthonormal bases in \mathbb{C}^2 and \mathbb{C}^m , respectively. One finds $4(m - 2)$ independent vectors taking $a \in \mathcal{B}_+(\mathbb{C}^2)$ arbitrary and $|h\rangle = \sum_{j=3}^m h_j f_j$. Now, we look for the remaining vectors $a \otimes |h\rangle$, with $|h\rangle = h_1 f_1 + h_2 f_2$. It is clear that it is enough to consider $a \in \mathcal{B}_+(\mathbb{C}^n)$ being rank-1 projector, i.e. $a = |x\rangle\langle x|$. One has

$$\Phi(|x\rangle\langle x|)|h\rangle = V^\dagger |\bar{x}\rangle\langle \bar{x}| V |h\rangle = \left(\sum_{i=1}^2 x_i h_i \right) \sum_{j=1}^2 \bar{x}_j |f_j\rangle. \quad (27)$$

Note that $\Phi(|x\rangle\langle x|)|h\rangle = 0$ for $|x\rangle \neq 0$ if and only if $\sum_{i=1}^2 x_i h_i = 0$, and hence (up to trivial scaling) $x_1 = h_2$ and $x_2 = -h_1$. The family of vectors $|x\rangle\langle x| \otimes |h\rangle \in \mathcal{B}(\mathbb{C}^2) \otimes \mathbb{C}^m$ is linearly independent iff the corresponding vectors $|\bar{x}\rangle \otimes |x\rangle \otimes |h\rangle$ are linearly independent in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^m$. Note that coordinates of $\bar{x} \otimes x \otimes h$ are polynomial functions of h_k and \bar{h}_k , namely:

$$h_1 h_2 \bar{h}_2, h_1 h_2 \bar{h}_1, h_1^2 \bar{h}_2, h_1 h_2 \bar{h}_2, h_2^2 \bar{h}_2, h_2^2 \bar{h}_1, h_2 h_1 \bar{h}_2, h_2^2 \bar{h}_2.$$

Note, that 6 of them are (functionally) linearly independent and hence one has 6 additional vectors $a \otimes |h\rangle$. Altogether, there are $4(m - 2) + 6 = 4m - 2$ linearly independent vectors.

(3 \Rightarrow 1) Follows from Theorem 3.2. \square

A similar problem was analyzed in [27] in the context of optimal decomposable maps. Recall that Φ is decomposable if $\Phi = \Phi_1 + \Phi_2 \circ \text{t}$, where Φ_1 and Φ_2 are completely positive. Equivalently, the corresponding entanglement witness W is decomposable if $W = Q_1 + (\mathbb{1}_{\mathcal{H}} \otimes \text{t})Q_2$, where $Q_1, Q_2 \in \mathcal{B}_+(\mathcal{H} \otimes \mathcal{K})$. Let us recall that $S \subset \mathcal{H} \otimes \mathcal{K}$ is a *completely entangled subspace* (CES) iff there is no nonzero product vectors in S . The authors of [27] proved the following

Theorem 4.2 *Let $\Phi : \mathcal{B}(\mathbb{C}^2) \longrightarrow \mathcal{B}(\mathbb{C}^m)$ be a positive decomposable map. The following conditions are equivalent*

1. Φ is optimal,

2. $\Phi(a) = \text{Tr}_{\mathbb{C}^2}(W a^t \otimes \mathbb{I}_m)$, where $W = (\mathbb{I}_2 \otimes t)Q$ and $Q \geq 0$ is supported on a CES,

3. P_Φ spans $\mathbb{C}^2 \otimes \mathbb{C}^m$.

Note, that we replaced optimality by exposedness, an arbitrary CES by a 1-dimensional CES supporting a positive operator

$$Q = \sum_{i,j=1}^2 |i\rangle\langle j| \otimes V^t |i\rangle\langle j| \overline{V} ,$$

with $\text{rank}(V) = 2$ (clearly, if $\text{rank}(V) = 1$, then Q is no longer supported on a CES). Finally, we replaced *weak* spanning property

$$\dim \text{span}_{\mathbb{C}} \{ |\bar{x}\rangle \otimes |h\rangle : \Phi(|x\rangle\langle x|)|h\rangle = 0 \} = 2m ,$$

by much stronger property (*strong spanning*)

$$\text{rank } \Phi(\mathbb{I}_2) + \dim \text{span}_{\mathbb{C}} \{ |\bar{x}\rangle \otimes |x\rangle \otimes |h\rangle : \Phi(|x\rangle\langle x|)|h\rangle = 0 \} = 4m .$$

5 A class of exposed decomposable maps $\mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^m)$

It was already shown by Marciniak [21] that all extremal decomposable maps $\mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^m)$ are exposed, i.e. maps of the form $\Phi(a) = V^\dagger a V$ and $\Phi(a) = V^\dagger a^t V$ are exposed. Now we show that being exposed these maps in general do not satisfy the *strong spanning property* (18).

Proposition 5.1 *Consider a positive decomposable map $\Phi : \mathcal{B}(\mathbb{C}^n) \rightarrow \mathcal{B}(\mathbb{C}^m)$ defined by $\Phi(a) = V^\dagger a^t V$. One has*

$$\dim N_\Phi = \begin{cases} m(n^2 - 1) & , \text{ rank}(V) > 1 \\ mn^2 - (2m - 1) & , \text{ rank}(V) = 1 \end{cases} \quad (28)$$

Proof: it is clear that it is enough to consider $a \in \mathcal{B}(\mathbb{C}^n)_+$ being rank-1 projector, i.e. $a = |x\rangle\langle x|$. Note, that using a linear transformation one can transform V to the following form $V = \sum_{i=1}^r |e_i\rangle\langle f_i|$, where $\{e_i\}_{i=1}^n, \{f_j\}_{j=1}^m$ are orthonormal bases in \mathbb{C}^n and \mathbb{C}^m , respectively.

Let $|\tilde{x}\rangle$ and $|\tilde{h}\rangle$ be vectors in \mathbb{C}^r built from the first r coordinates of $|x\rangle$ and $|h\rangle$, respectively. For a given vector $|x\rangle$, the orthogonal complement of $|\tilde{x}\rangle$ is spanned by $r - 1$ vectors

$$v_2 = | -x_2, x_1, 0, \dots, 0 \rangle , \quad v_3 = | -x_2, 0, x_1, 0, \dots, 0 \rangle , \quad \dots , \quad v_r = | -x_r, \dots, x_1 \rangle .$$

The general vector $|h\rangle$ orthogonal to $|x\rangle$ is then of the form $\sum_{i=2}^r \alpha_i |v_i\rangle \oplus |\hat{h}_i\rangle$ (where $\sum_{i=2}^r \alpha_i |\hat{h}_i\rangle = |h_{r+1}, \dots, h_m\rangle$). Observe, that $|h_{r+1}, \dots, h_m\rangle$ can be arbitrary. Now, a general vector $|h\rangle$ which is orthogonal to $|x\rangle$ is a linear combination of vectors from $r - 1$ subspaces:

$$\begin{aligned} H_2(x) &= \text{span}_{\mathbb{C}} \{ | -x_2, x_1, 0, \dots, 0 \rangle \} \oplus \mathbb{C}^{m-r} , \\ H_3(x) &= \text{span}_{\mathbb{C}} \{ | -x_3, 0, x_1, 0, \dots, 0 \rangle \} \oplus \mathbb{C}^{m-r} , \\ &\vdots \\ H_r(x) &= \text{span}_{\mathbb{C}} \{ | -x_r, 0, \dots, 0, x_1 \rangle \} \oplus \mathbb{C}^{m-r} . \end{aligned}$$

Consider the subspace $W_2 \subset \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m \cong \mathcal{B}(\mathbb{C}^n) \otimes \mathbb{C}^m$ spanned by the vectors $|\bar{x}\rangle \otimes |x\rangle \otimes |h\rangle$, where $|h\rangle \in H_2(x)$, that is, $|\bar{x}\rangle \otimes |y\rangle$, where

$$|y\rangle = |x_1, \dots, x_r, x_{r+1}, \dots, x_n\rangle \otimes |-x_2, x_1, 0, \dots, 0, h_{r+1}, \dots, h_m\rangle = \sum_{i,j} y_{ij} e_i \otimes f_j. \quad (29)$$

The coordinates of $|y\rangle$ are monomials of degree 2 in variables $\{x_1, \dots, x_n, h_{r+1}, \dots, h_m\}$. Note that $|y\rangle$ has in general $(2 + m - r) \times n$ non-zero coordinates, which satisfy one linear condition $y_{11} + y_{22} = 0$. Hence $\dim W_2 = n(n[m - r + 2] - 1)$. Using the same argument one shows that $\dim W_2 = \dots = \dim W_r$. It is easy to show that

$$W_i \cap W_j = W_2 \cap \dots \cap W_r, \quad (30)$$

for each pair $i \neq j$. Moreover, the constructions of $H_i(x)$ imply

$$W_2 \cap \dots \cap W_r = \mathbb{C}^n \otimes (\text{span}_{\mathbb{C}}\{e_2, \dots, e_n\} \otimes f_1 \oplus \mathbb{C}^n \otimes \text{span}_{\mathbb{C}}\{f_{r+1}, \dots, f_m\}),$$

and hence its dimension equals $n(n - 1 + [m - r]n)$. Let $W = \text{span}_{\mathbb{C}}(W_2 \cup \dots \cup W_r)$. One finds

$$\begin{aligned} \dim W &= \sum_{i=2}^r \dim W_i - (r - 2) \cdot \dim(W_2 \cap \dots \cap W_r) \\ &= (r - 1)n((m - r + 2)n - 1) - (r - 2)n(n - 1 + (m - r)n) = n^2m - n. \end{aligned}$$

Note that if $r = 1$, one consider vectors $|\bar{x}\rangle \otimes |x\rangle \otimes |h\rangle$ such that $x_1 h_1 = 0$. Vectors with $x_1 = 0$ form a $(n - 1)^2 m$ dimensional subspace. Vectors with $h_1 = 0$ form a $n^2(m - 1)$ dimensional subspace. The intersection of these subspaces is $(n - 1)^2(m - 1)$ dimensional. Finally, one gets $(n - 1)^2 m + n^2(m - 1) - (n - 1)^2(m - 1) = n^2m - (2n - 1)$ linearly independent vectors. \square

It is therefore clear that the *strong spanning property*

$$\text{rank } \Phi(\mathbb{I}_n) + \dim \text{span}_{\mathbb{C}} \{ |\bar{x}\rangle \otimes |x\rangle \otimes |h\rangle : \Phi(|x\rangle\langle x|)|h\rangle = 0 \} = mn^2,$$

supplemented by irreducibility provides only a sufficient condition for exposedness in the same way as *weak spanning property*

$$\dim \text{span}_{\mathbb{C}} \{ |\bar{x}\rangle \otimes |h\rangle : \Phi(|x\rangle\langle x|)|h\rangle = 0 \} = nm,$$

provides only a sufficient condition for optimality. Note, that $\Phi(a) = V^\dagger a^\dagger V$ has a *strong spanning property* iff $\text{rank}(V) = n$. However, Φ is exposed for any V [21].

6 Conclusions

We have provided a sufficient condition for exposedness – *strong spanning property* (18). It was shown that in the class of decomposable maps $\mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^m)$ this condition is also necessary if $n = 2$. This result provides an analog of the result of [27] in the context of optimal maps/witnesses. One calls a block-positive operator $W \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^m)$ irreducible iff W cannot be written as $W_1 \oplus W_2$, where W_1 and W_2 are block-positive. One has the following

Proposition 6.1 *Let $W \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^m)$ be a block-positive irreducible operator. If*

$$\dim(\text{Im Tr}_{\mathbb{C}^n} W) + \dim\{a \otimes h : \text{Tr}_{\mathbb{C}^n}[W a^t \otimes \mathbb{I}_m]|h\rangle = 0\} = n^2 m, \quad (31)$$

then W is exposed.

If $n = 2$, then one proves the following

Proposition 6.2 *Let $W \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^m)$ be a block-positive but not positive decomposable operator (i.e. decomposable entanglement witness). The following conditions are equivalent*

1. *W is exposed,*
2. *$W = (\mathbb{I}_2 \otimes t)Q$, and Q is Schmidt rank 2 projector,*
3. *There are $3m$ linearly independent vectors $|\bar{x}\rangle\langle\bar{x}| \otimes |h\rangle \in \mathcal{B}_+(\mathbb{C}^2 \otimes \mathbb{C}^n)$ such that*

$$\langle x \otimes h | W | x \otimes h \rangle = 0.$$

In the forthcoming paper we use the *strong spanning property* to analyze exposed positive indecomposable maps.

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